C4 Exercise 6D (integration with partial fractions)

Note Title 04/07/2007

example

find $\displaystyle\int \frac{x-5}{(x+1)(x-2)}\,dx$

$$\frac{x-5}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$\Rightarrow \quad x-5 = A(x-2) + B(x+1)$$

put $x = 2 \Rightarrow -3 = 3B \Rightarrow B = -1$

put $x = -1 \Rightarrow -6 = -3A \Rightarrow A = 2$

then $\displaystyle\int \frac{x-5}{(x+1)(x-2)}\,dx \equiv \int \left(\frac{2}{x+1} - \frac{1}{x-2}\right) dx$

$$= 2\ln|x+1| - \ln|x-2| + C$$

C4Ex 6D

example ctd:

Now this final answer

$$2\ln|x+1| - \ln|x-2| + c$$

can be written as a single logarithm:

$$\equiv \ln|(x+1)^2| + \ln|(x-2)^{-1}| + C$$

$$\equiv \ln\left|\frac{(x+1)^2}{x-2}\right| + C$$

C4Ex6D

example

find $\int \frac{8x^2 - 19x + 1}{(2x+1)(x-2)^2}\,dx = I$ ← call the integral I.

$$\frac{8x^2 - 19x + 1}{(2x+1)(x-2)^2} \equiv \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\Rightarrow \quad 8x^2 - 19x + 1 \equiv A(x-2)^2 + B(2x+1)(x-2) + C(2x+1)$$

put $x = -\frac{1}{2}$ $\Rightarrow$ $2 + 9\frac{1}{2} + 1 = \frac{25}{4}A$

$$\Rightarrow \quad A = \frac{12\frac{1}{2} \times 4}{25} = 2$$

put $x = 2$ $\Rightarrow$ $32 - 38 + 1 = 5C$

$$\Rightarrow \quad C = -1$$

C4 Ex 6D

example ctd: equate coefficients of x^2

$$8 = A + 2B$$

$$\Rightarrow \quad 2B = 6$$

$$B = 3$$

hence

$$I = \int \left[\frac{2}{2x+1} + \frac{3}{x-2} + \frac{-1}{(x-2)^2} \right] dx$$

$$I = 2 \times \frac{1}{2} \ln|2x+1| + 3 \ln|x-2| + (x-2)^{-1} + C$$

$$= \ln\left|(2x+1)(x-2)^3\right| + \frac{1}{x-2} + C$$

C4Ex6D

example

find $\int \frac{2}{(1-x^2)}\,dx = I$

note $1-x^2 \equiv (1+x)(1-x)$

so write $\frac{2}{1-x^2} \equiv \frac{A}{1+x} + \frac{B}{1-x}$

$\Rightarrow \quad 2 \equiv A(1-x) + B(1+x)$

put $x=1 \Rightarrow B=1$

put $x=-1 \Rightarrow A=1$

$\Rightarrow \quad I = \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx$

C4 Ex 6D

example ctd.

$$I = \ln|1+x| - \ln|1-x| + C$$

$$= \ln\left|\frac{1+x}{1-x}\right| + C$$

C4 Ex6D

example

find $\displaystyle\int \frac{9x^2 - 3x + 2}{9x^2 - 4}\,dx = I$

this is an improper fraction so we'll need to divide first:

$$
\begin{array}{r|llll}
 & 1 & & & \\ \hline
9x^2 + 0x - 4 & 9x^2 & -3x & +2 & \\
 & -(9x & +0x & -4) & \\ \hline
 & & -3x & +6 &
\end{array}
$$

$$\therefore\ I = \int \left(1 + \frac{-3x+6}{9x^2-4} \right) dx$$

C4 Ex 6D

example ctd $\quad I = \int 1 + \dfrac{-3x+6}{(3x-2)(3x+2)} \, dx$

now $\quad \dfrac{-3x+6}{(3x-2)(3x+2)} = \dfrac{A}{3x-2} + \dfrac{B}{3x+2}$

$\Rightarrow \quad -3x + 6 = A(3x+2) + B(3x-2)$

putting $x = 2/3$ $\quad 4 = 4A \Rightarrow A = 1$

putting $x = -2/3$ $\quad 8 = -4B \Rightarrow B = -2$

so $\quad I = \int \left(1 + \dfrac{1}{3x-2} - \dfrac{2}{3x+2}\right) dx$

example ctd

hence $$I = x + \frac{1}{3}\ln|3x-2| - \frac{2}{3}\ln|3x+2| + C$$

$$I = x + \frac{1}{3}\ln\left|\frac{3x-2}{(3x+2)^2}\right| + C$$

END OF EXAMPLES

1a $\displaystyle\int \frac{3x+5}{(x+1)(x+2)}\,dx$

$$\frac{3x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\Rightarrow \quad 3x+5 = A(x+2) + B(x+1)$$

put $x = -1$

$$2 = A$$

put $x = -2$

$$-1 = -B$$

$$B = 1$$

1a (ctd)

$$\int \frac{3x+5}{(x+1)(x+2)}\,dx \equiv \int \left(\frac{2}{x+1} + \frac{1}{x+2} \right) dx$$

$$= 2\ln|x+1| + \ln|x+2| + C$$

$$= \ln\left|(x+1)^2(x+2)\right| + C$$

16 $\displaystyle\int \frac{3x-1}{(2x+1)(x-2)}\,dx$

$$\frac{3x-1}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2}$$

$$\Rightarrow \quad 3x-1 = A(x-2) + B(2x+1)$$

put $x = -1/2$

$$-5/2 = -5/2\,A$$

$$\Rightarrow \quad A = 1$$

put $x = 2$

$$5 = 5B \quad \Rightarrow \quad B = 1$$

1b $\displaystyle\int \frac{3x - 1}{(2x+1)(x-2)}\, dx \equiv \int \left(\frac{1}{(2x+1)} + \frac{1}{(x-2)} \right) dx$

$$\equiv \int \left[\frac{1}{2x+1} + \frac{1}{x-2} \right] dx$$

$$= \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$$

$$= \ln\left|(x-2)\sqrt{2x+1}\right| + C$$

1c $\int \frac{2x-6}{(x+3)(x-1)}\,dx$ $\qquad \frac{2x-6}{(x+3)(x-1)} \equiv \frac{A}{x+3} + \frac{B}{x-1}$

$$2x-6 = A(x-1) + B(x+3)$$

put $x=-3 \Rightarrow -12 = -4A \Rightarrow A = 3$

put $x = 1 \Rightarrow -4 = 4B \Rightarrow B = -1$

$$\int \frac{2x-6}{(x+3)(x-1)}\,dx \equiv \int \left(\frac{3}{x+3} - \frac{1}{x-1}\right) dx$$

$$= 3\ln|x+3| - \ln|x-1| + C$$

$$= \ln\left|\frac{(x+3)^3}{x-1}\right| + C$$

C4 Ex 6D

1d $\displaystyle\int \frac{3}{(2+x)(1-x)}\,dx$

$$\frac{3}{(2+x)(1-x)} \equiv \frac{A}{2+x} + \frac{B}{1-x}$$

$$3 = A(1-x) + B(2+x)$$

put $x = -2 \Rightarrow 3 = 3A \Rightarrow A = 1$

put $x = 1 \Rightarrow 3 = 3B \Rightarrow B = 1$

$$\int \frac{3}{(2+x)(1-x)}\,dx = \int \left(\frac{1}{2+x} + \frac{1}{1-x}\right) dx$$

$$= \ln|2+x| - \ln|1-x| + C$$

$$= \ln\left|\frac{2+x}{1-x}\right| + C$$

C4 Ex 6D

1e $\displaystyle\int \frac{4}{(2x+1)(1-2x)}\,dx$

$$\frac{4}{(2x+1)(1-2x)} \equiv \frac{A}{(2x+1)} + \frac{B}{(1-2x)}$$

$$\Rightarrow \quad 4 = A(1-2x) + B(2x+1)$$

put $x = \frac{1}{2}$ $\Rightarrow$ $4 = 2B$ $\Rightarrow$ $B = 2$

put $x = -\frac{1}{2}$ $\Rightarrow$ $4 = 2A$ $\Rightarrow$ $A = 2$

$$\int \frac{4}{(2x+1)(1-2x)}\,dx = \int \left(\frac{2}{2x+1} + \frac{2}{1-2x}\right)dx = \frac{2}{2}\ln|2x+1| + \frac{2}{-2}\ln|1-2x| + C$$

$$= \ln\left|\frac{2x+1}{1-2x}\right| + C$$

C4 Ex 6D

1f
$$\int \frac{3(x+1)}{9x^2-1}\,dx \equiv \int \frac{3x+3}{(3x+1)(3x-1)}\,dx$$

$$\frac{3x+3}{(3x+1)(3x-1)} \equiv \frac{A}{3x+1} + \frac{B}{3x-1}$$

$$3x+3 = A(3x-1) + B(3x+1)$$

put $x = \frac{1}{3} \Rightarrow \quad 4 = 2B \quad \Rightarrow \quad B = 2$

put $x = -\frac{1}{3} \Rightarrow \quad 2 = -2A \quad \Rightarrow \quad A = -1$

$$\int \frac{3(x+1)}{9x^2-1}\,dx \equiv \int \left[\frac{-1}{3x+1} + \frac{2}{3x-1} \right] dx$$

$$= \frac{-1}{3}\ln|3x+1| + \frac{2}{3}\ln|3x-1| + C$$

$$= \frac{1}{3}\ln\left|\frac{(3x-1)^2}{3x+1}\right| + C$$

C4 Ex 6D

1g $I = \int \frac{3-5x}{(1-x)(2-3x)}\,dx$

$$\frac{3-5x}{(1-x)(2-3x)} = \frac{A}{(1-x)} + \frac{B}{(2-3x)}$$

$$\Rightarrow \quad 3-5x = A(2-3x) + B(1-x)$$

putting $x = 1 \Rightarrow -2 = -A \Rightarrow A = 2$

putting $x = 2/3 \Rightarrow -1/3 = 1/3\,B \Rightarrow B = -1$

$$I = \int \left(\frac{2}{1-x} + \frac{-1}{2-3x} \right) dx = \frac{2}{-1} \ln|1-x| + \frac{-1}{-3} \ln|2-3x| + C$$

$$I = \ln\left| \frac{(2-3x)^{1/3}}{(1-x)^2} \right| + C$$

C4 Ex 6D

1h $\quad I = \displaystyle\int \frac{x^2-3}{(2+x)(1+x)^2}\,dx$

$$\frac{x^2-3}{(2+x)(1+x)^2} \equiv \frac{A}{2+x} + \frac{B}{(1+x)^2} + \frac{C}{1+x}$$

$$\Rightarrow \quad x^2-3 = A(1+x)^2 + B(2+x) + C(1+x)(2+x)$$

put $x=-1 \Rightarrow 1-3 = 0 + B + 0 \Rightarrow B = -2$

put $x=-2 \Rightarrow 4-3 = A + 0 + 0 \Rightarrow A = 1$

now equate constants $-3 = A + 2B + 2C \Rightarrow 2C = -3-1+4 \Rightarrow C=0$

so $I \equiv \displaystyle\int \frac{x^2-3}{(2+x)(1+x)^2}\,dx \equiv \int \frac{1}{2+x} - \frac{2}{(1+x)^2}\,dx$

$$= \ln|2+x| - \frac{2}{1+x} + C$$

C4 Ex 6D

1i $\quad I = \displaystyle\int \frac{5+3x}{(x+2)(x+1)^2}\,dx$

$$\frac{5+3x}{(x+2)(x+1)^2} \equiv \frac{A}{x+2} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$$

$\Rightarrow \quad 5+3x = A(x+1)^2 + B(x+2) + C(x+1)(x+2)$

put $x=-2 \Rightarrow -1 = A$

put $x=-1 \Rightarrow 2 = B$

equate constant terms $\quad 5 = A + 2B + 2C \Rightarrow 5 = -1 + 4 + 2c \Rightarrow c = 1$

$$I = \int \frac{5+3x}{(x+2)(x+1)^2}\,dx = \int \left[\frac{-1}{x+2} + \frac{2}{(x+1)^2} + \frac{1}{x+1}\right] dx$$

$$I = -\ln|x+2| + \frac{-2}{x+1} + \ln|x+1| = \ln\left|\frac{x+1}{x+2}\right| - \frac{2}{x+1} + C$$

$* \displaystyle\int \frac{2}{y^2}\,dy = \int 2y^{-2}\,dy = \frac{2}{-1}y^{-1} + C = -2y^{-1} + C$

C4Ex6D

1j $I = \int \frac{17-5x}{(3+2x)(2-x)^2}\,dx = \int \left[\frac{A}{3+2x} + \frac{B}{(2-x)^2} + \frac{C}{2-x}\right] dx$

since $17 - 5x = A(2-x)^2 + B(3+2x) + C(3+2x)(2-x)$

put $x = -3/2 \Rightarrow 17 + 15/2 = A\left(7/2\right)^2 \Rightarrow \frac{49}{2} = \frac{49}{4}A \Rightarrow A = 2$

put $x = 2 \Rightarrow 17 - 10 = B(7) \Rightarrow B = 1$

equate constants $17 = 4A + 3B + 6C \Rightarrow 6C = 17 - 8 - 3 = 6$

$\Rightarrow C = 1$

$I = \int \left[\frac{2}{3+2x} + \frac{1}{(2-x)^2} + \frac{1}{2-x}\right] dx = \frac{2}{2}\ln|3+2x| + \frac{1}{-1}\frac{(2-x)^{-1}}{-1} + \frac{1}{-1}\ln|2-x| + C$

$I = \frac{1}{2-x} + \ln\left|\frac{3+2x}{2-x}\right| + C$

rule 10 (p85)

$\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$

C4 Ex 6D

2a

$$I \equiv \int \frac{2(x^2+3x-1)}{(x+1)(2x-1)}\,dx \equiv \int \frac{2x^2+6x-2}{2x+x-1}\,dx$$

now since $\quad 2x+x-1 \overline{\big)\, 2x^2+6x-2}$ with quotient 1

$$\begin{array}{r} 2x^2+6x-2 \\ -\,(2x^2+x-1) \\ \hline 5x-1 \end{array}$$

, $\quad I \equiv \int \left[1 + \frac{5x-1}{(x+1)(2x-1)}\right] dx$

$$\frac{5x-1}{(x+1)(2x-1)} \equiv \frac{A}{x+1} + \frac{B}{2x-1} \Rightarrow 5x-1 = A(2x-1) + B(x+1)$$

put $x=-1 \Rightarrow -6 = A(-3) \Rightarrow A=2$

put $x = \frac{1}{2} \Rightarrow \frac{3}{2} = \frac{3}{2}B \Rightarrow B=1$

$$I \equiv \int \left[1 + \frac{2}{x+1} + \frac{1}{2x-1}\right] dx = x + 2\ln|x+1| + \frac{1}{2}\ln|2x-1| + C$$

$$= x + \ln\left|(x+1)^2\sqrt{2x-1}\right| + C$$

C4Ex6D

2b $\displaystyle\int \frac{x^3+2x^2+2}{x(x+1)}\,dx \equiv \int \left(x+1+\frac{2-x}{x(x+1)}\right)dx$

Since

$$\begin{array}{r|l} & x+1 \\ \hline x^2+x+0 & x^3+2x^2+0x+2 \\ & -(x^3+x^2+0x) \\ \hline & x^2+0x+2 \\ & -(x^2+1x+0) \\ \hline & -x+2 \end{array}$$

and $\dfrac{2-x}{x(x+1)} \equiv \dfrac{A}{x}+\dfrac{B}{x+1}$

$\Rightarrow\ 2-x = A(x+1)+Bx$

$\Rightarrow\ A=2$ and $B=-3$

so $\displaystyle\int \frac{x^3+2x^2+2}{x(x+1)}\,dx \equiv \int \left[x+1+\frac{2}{x}-\frac{3}{x+1}\right]dx$

$$= \tfrac{1}{2}x^2+x+2\ln|x|-3\ln|x+1|+C$$

$$= \frac{x^2}{2}+x+\ln\left|\frac{x^2}{(x+1)^3}\right|+C$$

C4Ex6D

2c $I = \int \frac{x^2}{x^2-4}\,dx$

$$\begin{array}{r} 1 \\ x^2+0x-4 \overline{) x^2+0x+0} \\ -(x^2+0x-4) \\ \hline 4 \end{array}$$

so $I = \int 1 + \frac{4}{(x+2)(x-2)}\,dx$

$$\frac{4}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} \Rightarrow 4 = A(x-2) + B(x+2)$$

$x=2 \Rightarrow B=1$

$x=-2 \Rightarrow A=-1$

$$I = \int \left[1 - \frac{1}{x+2} + \frac{1}{x-2}\right] dx = x - \ln|x+2| + \ln|x-2| + C$$

$$= x + \ln\left|\frac{x-2}{x+2}\right| + C$$

C4 Ex6D

2d $\displaystyle\int \frac{x^2+x+2}{3-2x-x^2}\,dx \equiv \int \left[-1 + \frac{5-x}{(3+x)(1-x)}\right] dx$

since

$$\begin{array}{r} -1 \\ -x^2-2x+3\,\big)\overline{\,x^2+x+2\,} \\ -(x^2+2x-3) \\ \hline -x+5 \end{array}$$

and $\displaystyle\frac{5-x}{(3+x)(1-x)} = \frac{A}{3+x} + \frac{B}{1-x}$

now if $5-x = A(1-x) + B(3+x)$ put $x=1 \Rightarrow 4 = 4B \Rightarrow B=1$

and if we put $x=-3$, $8 = 4A \Rightarrow A=2$

so $\displaystyle\int \frac{x^2+x+2}{3-2x-x^2}\,dx \equiv \int \left[-1 + \frac{2}{3+x} + \frac{1}{1-x}\right] dx$

watch out $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|\;|$

$= -x + 2\ln|3+x| + \frac{1}{-1}\ln|1-x| + C$

$= -x + \ln\left|\frac{(3+x)^2}{1-x}\right| + C$

C4Ex6D

2e $\displaystyle\int \frac{6+3x-x^2}{x^3+2x^2}\,dx \equiv \int \frac{6+3x-x^2}{x^2(x+2)}\,dx$

$$\frac{6+3x-x^2}{x^2(x+2)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+2}$$

$\Rightarrow \quad 6+3x-x^2 = A(x+2) + Bx(x+2) + Cx^2$

put $x=0$ (and equate constant terms) $\quad 6 = 2A \Rightarrow A = 3$

put $x=-2 \quad \cancel{6}-\cancel{6}-(-2)^2 = 0+0+C(-2)^2 \Rightarrow C=-1$

equate coefficients of x^2: $\quad -1 = B + C \Rightarrow B = 0$

so $\displaystyle\int \frac{6+3x-x^2}{x^3+2x^2}\,dx \equiv \int\left[\frac{3}{x^2} - \frac{1}{x+2}\right]dx = -3x^{-1} - \ln|x+2| + C$

$$= \frac{-3}{x} - \ln|x+2| + C$$